On the Gaussian Free Field and on its thick points

Clara Antonucci[1]

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[1]This file contains the notes that I have written for a seminar about the Gaussian Free Field, prepared under the supervision of Prof. Franco Flandoli. The content of this work is based on the article "Thick points of the Gaussian Free Field" ([2]).

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The Gaussian Free Field and its thick points

1.1 Heuristic and physical motivations

The aim of this chapter is the definition of the Gaussian Free Field, which can be seen as the analogue of the Brownian Motion in which the time is d-dimensional. In this chapter we consider the d-dimensional torus T^d as domain and we always work with the Periodic Boundary Conditions, therefore every Sobolev space of functions of the torus will implicitly be restricted to zero-mean functions.

Physical reasons suggest that $d = 2$ is more relevant than other dimensions, and the definition of thick points requires explicitly to work only in dimension two.

At the end of this chapter, we spend a few words about the construction of the Gaussian Free Field for other domains (in particular regular bounded subsets of the plane). Such constructions are a bit more technical, but the idea is still quite intuitive.

1.2 Definition of the Gaussian Free Field in the torus

We consider the following family of $L^2(T^d)$

$$\left\{e^{2\pi i\langle k,x\rangle}\right\}_{k\in\mathbb{Z}_0^d},$$

where $\mathbb{Z}_0^d := \mathbb{Z}^d \setminus \{0\}$. Since we are considering only zero-mean functions, this set is an orthonormal basis of $L^2(T^d)$.

Let us consider a countable family of i.i.d. standard Gaussian random variables

$$\{G_k : (\Omega, \mathcal{F}, \mathcal{P}) \to \mathbb{R}\}_{k\in\mathbb{Z}_0^d}.$$

It is possible to consider, at least formally, the series

$$X(\omega) := \sum_{k\in\mathbb{Z}_0^d} |k|^{-1} G_k(\omega) e_k. \tag{1.2.1}$$

The previous formula can easily be rewritten as

$$X(\omega) := \sum_{k\in\mathbb{Z}_0^d} \sigma_k G_k(\omega) f_k, \tag{1.2.2}$$

where

$$\begin{cases} \alpha \in (0, +\infty)\,, \\ \sigma_k := |k|^{-\alpha}\,, \\ f_k := |k|^{\alpha-1}\, e_k. \end{cases}$$

We recall that f_k is an orthonormal basis of the Sobolev space $W^{-\alpha+1,2}(T^d)$.

Proposition 1.2.1. Le us define X_N as the partial sums of (1.2.2), namely

$$X_N(\omega) := \sum_{k \in \mathbb{Z}_0^d,\ |k|<N} \sigma_k G_k(\omega) f_k.$$

Then the random variables

$$X_N : (\Omega, \mathcal{F}, \mathcal{P}) \to W^{-\alpha+1,2}(T^d)$$

are a Cauchy sequence in the space $L^2(\Omega, W^{-\alpha+1,2}(T^d))$ if and only if the series

$$\sum_{k \in \mathbb{Z}_0^d} \sigma_k^2$$

converges.

Definition 1.2.2. If the hypothesis

$$\sum_{k \in \mathbb{Z}_0^d} \sigma_k^2 < +\infty$$

is fulfilled, then the random variable X given by (1.2.2) is well defined as an element of $L^2(\Omega, W^{-\alpha+1,2}(T^d))$.

Definition 1.2.3 (Gaussian Free Field)**.** Since the series

$$\sum_{k \in \mathbb{Z}_0^d} |k|^{-2\alpha}$$

converges if and only if $\alpha > \frac{d}{2}$, Proposition 1.2.1 implies that for every $\alpha > \frac{d}{2}$ the random variable

$$F := \sum_{k \in \mathbb{Z}_0^d} \frac{1}{|k|} G_k e_k = \sum_{k \in \mathbb{Z}_0^d} |k|^{-\alpha}\, G_k f_k \tag{1.2.3}$$

is well defined as an $L^2(\Omega, H^{-\alpha+1,2}(T^d))$ limit.

The random variable F in $H^{-\frac{d}{2}+1-}(T^d)$ is called Gaussian Free Field.

1.3 Main properties of the Gaussian Free Field

1.3.1 Integrability properties

Since the Gaussian Free Field is defined as an $L^2(\Omega, V)$ equivalence class (where V is the suitable Sobolev space), it makes no sense in general to consider $F(\omega)$ for a fixed value of ω. In fact changing the value of F on a negligible subset of Ω does not change the random variable as an element of $L^2(\Omega, V)$.

Another aspect that underlines the fact that the correct way to look at F is as an element of $L^2(\Omega, V)$ and not pathwise is the integrability. If we consider the random variable for a fixed value of ω, all that we can say is that

$$F(\omega) \in H^{-\frac{d}{2}+1-}(T^d).$$

In particular the duality $\langle F(\omega), g\rangle$ is well defined if and only if

$$g \in H^{\frac{d}{2}-1+\varepsilon}(T^d).$$

The following theorem, however, shows that randomness allows to improve integrability properties.

Theorem 1.3.1. Let us consider any element $g \in H^{-1}(T^d)$. Then the sequence

$$\langle F_N, g\rangle : \Omega \to \mathbb{R}$$

is a Cauchy sequence in $L^2(\Omega, \mathbb{R})$. Therefore for every $g \in H^{-1}(T^d)$ it is well defined its limit

$$\langle F, g\rangle \in L^2(\Omega, \mathbb{R}).$$

It is a real Gaussian random variable with law of $N(0, \|g\|^2_{H^{-1}})$.

In an analogous way it is possible to prove the following Proposition.

Proposition 1.3.2.

$$\mathbb{E}\left[\langle F, g_1\rangle \langle F, g_2\rangle\right] = \langle g_1, g_2\rangle_{H^{-1}} = \left\langle \Delta^{-1} g_1, g_2\right\rangle = \iint G(x,y) g_1(x) g_2(y).$$

This Proposition justifies formally the intuitive concept that the correlation of the Gaussian Free Field is the Green kernel.

1.3.2 Averages of the Gaussian Free Field

As we have seen, the Gaussian Free Field is no more than a distribution (although in dimension $d = 2$ it takes values in $H^{-}(T)$), therefore it is not well defined pointwise. However, as soon as we consider certain kinds of averages, it becomes a proper function. There are basically two ways of taking averages.

The first one is to consider a borelian set $A \subseteq T^d$. The indicator function $\mathbb{1}_A$ is an element of $L^2(T^d)$. Therefore, if $|A| > 0$, it is well defined

$$\left\langle F, \frac{1}{|A|}\mathbb{1}_A \right\rangle \in L^2(\Omega, \mathbb{R}).$$

This allows us to speak about averages of the Gaussian Free Field on borelian sets, defined as

$$\frac{1}{|A|}\int_A F(x)dx := \left\langle F, \frac{1}{|A|}\mathbb{1}_A \right\rangle.$$

Alternatively, if we deal with the two dimensional torus, from now on denoted by T, given a simple smooth curve $\gamma : [0,1] \to T$, we can consider the function

$$\begin{aligned}\Gamma_\gamma : \ & H^1(T) \to \mathbb{R} \\ & f \mapsto \int_\gamma f(\sigma) d\sigma.\end{aligned}$$

It is possible to show that $\Gamma_\gamma \in H^{-1}(T)$, therefore we can define

$$\frac{1}{|\gamma|}\int_\gamma F(\sigma)d\sigma := \left\langle F, \frac{1}{|\gamma|}\Gamma_\gamma \right\rangle.$$

In the case in which $\gamma_{z,r}$ parametrizes $\partial D(z,r)$, this is equivalent to

$$F(z,r) := \frac{1}{|\gamma_{z,r}|}\int_{\gamma_{z,r}} F(\sigma)d\sigma = \langle F, \eta_{z,r}\rangle , \tag{1.3.1}$$

where $\eta(z,r)$ is the uniform measure on $\partial D(z,r)$. The duality is well defined, since $\eta_{z,r} \in H^{-1}(T)$.

Using Proposition 1.3.2 it is possible to compute the covariance of the process $F(z,r)$

$$\begin{aligned}\mathbb{E}\left[F(z,r)F(w,s)\right] &= \mathbb{E}\left[\langle F,\eta_{z,r}\rangle \langle F,\eta_{w,s}\rangle\right] \\ &= \langle \eta_{z,r}, \eta_{w,s}\rangle_{H^{-1}} \\ &= \left\langle (\Delta^{-1}\eta_{z,r}), \eta_{w,s}\right\rangle \\ &= \iint G(x,y)\,\mathrm{d}\eta_{z,r}(x)\,\mathrm{d}\eta_{w,s}(y). \end{aligned}\tag{1.3.2}$$

Given the expression (1.3.2) it is possible to show the following two facts:

- If we fix z and consider $t_1 := \log\left(\frac{1}{r}\right)$, $t_2 := \log\left(\frac{1}{s}\right)$, then

$$\mathbb{E}\left[F(z,e^{-t_1})F(z,e^{-t_2})\right] = \frac{\min\{t_1,t_2\}}{2\pi} + C(z).$$

  This comes from the fact that $G(x,y)$ can be written as $\log(|x-y|)$ plus a regular function and a computation of complex analysis. Therefore the process $\left\{\sqrt{2\pi}\left(F(z,e^{-t}) - F(z,e^{-t_1})\right)\right\}_{t\geq t_1}$ has the mean and the covariance function of a Brownian motion.

- This quantitative estimate holds

$$\left|\mathbb{E}\left[F(z,r)F(w,s)\right] - \mathbb{E}\left[F(z',r')F(w',s')\right]\right| \leq C\frac{|(z,r,w,s)-(z',r',w',s')|}{(r\wedge r')\vee(s\wedge s')}, \tag{1.3.3}$$

  where the right-hand is not bounded when one of the radii goes to zero because the variance of $F(z,r)$ is equal to $\frac{1}{2\pi}\log(\frac{1}{r})$. We do not report the computation, that can be found in [2].

What we need now is the following theorem.

Theorem 1.3.3 (Modified Kolmogorov-Centsov Theorem)**.** Let U be a bounded open set of $\mathbb{R}^d$ and X be a process from $U \times (0,1]$ to $\mathbb{R}$. If there exist some $\alpha, \beta > 0$ such that

$$\mathbb{E}\left[|X(z,r) - X(w,s)|^{\alpha}\right] \leq C(\alpha)\left(\frac{|(z,r)-(w,s)|}{r \wedge s}\right)^{d+1+\beta},$$

then for every $\zeta > \alpha^{-1}$, for every $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$ there exists a modification $\tilde{X}$ of X such that there exists a constant $M = M(\alpha, \beta, \gamma, \zeta)$ such that for every $z, w \in U$, for every $r, s \in (0,1]$ with $\frac{r}{s} \in \left[\frac{1}{2}, 2\right]$

$$\left|\tilde{X}(z,r) - \tilde{X}(w,s)\right| \leq M\left(\log\frac{1}{r}\right)^{\zeta}\left(\frac{|(z,r)-(w,s)|^{\gamma}}{r^{\tilde{\gamma}}}\right),$$

where $\tilde{\gamma} = \frac{d+\beta}{\alpha}$.

In our case, this theorem applies and the following result holds.

Theorem 1.3.4 (Existence of a Hölder-continuous version)**.** The process $F(z,r)$ admits a modification $\tilde{F}(z,r)$ such that

$$\text{for every } \gamma \in \left(0, \frac{1}{2}\right), \text{ for every } \delta, \zeta > 0$$
$$\text{there exists a constant } M = M(\gamma, \delta, \zeta) \text{ such that}$$
$$\text{for every } z, w \in U, \text{ for every } r, s \in (0,1] \text{ with } \frac{r}{s} \in \left[\frac{1}{2}, 2\right]$$
$$\left|\tilde{F}(z,r) - \tilde{F}(w,s)\right| \leq M\left(\log\left(\frac{1}{r}\right)\right)^{\zeta}\left(\frac{|(z,r)-(w,s)|^{\gamma}}{r^{\gamma+\delta}}\right). \tag{1.3.4}$$

Proof. We estimate

$$\begin{aligned}
\mathbb{E}\left[\Big(F(z,r) - F(w,s)\Big)^2\right] &= \mathbb{E}\left[F(z,r)^2\right] - \mathbb{E}\left[F(z,r)F(w,s)\right] + \mathbb{E}\left[F(w,s)^2\right] - \mathbb{E}\left[F(z,r)F(w,s)\right] \\
&\leq \left|\mathbb{E}\left[F(z,r)^2\right] - \mathbb{E}\left[F(z,r)F(w,s)\right]\right| + \left|\mathbb{E}\left[F(w,s)^2\right] - \mathbb{E}\left[F(z,r)F(w,s)\right]\right| \\
&\leq C\frac{|(z,r)-(w,s)|}{r \wedge s}, \qquad (1.3.5)
\end{aligned}$$

where for the last inequality we have used (1.3.3).

Now we recall that $F(z,r) - F(w,s)$ is a Gaussian random variable, therefore it is possible to estimate any moment with a suitable power of the second moment, namely

$$\mathbb{E}\left[\Big|F(z,r) - F(w,s)\Big|^{\alpha}\right] \leq K(\alpha) \cdot \mathbb{E}\left[\Big(F(z,r) - F(w,s)\Big)^2\right]^{\frac{\alpha}{2}}. \tag{1.3.6}$$

Therefore, combining (1.3.6) and (1.3.5) it follows that for every $\alpha > 1$,

$$\mathbb{E}\left[\Big|F(z,r) - F(w,s)\Big|^{\alpha}\right] \leq C(\alpha)\left(\frac{|(z,r)-(w,s)|}{r \wedge s}\right)^{\frac{\alpha}{2}}.$$

Now the thesis follows applying Theorem 1.3.3. $\square$

This modification is unique, since it is well know that two modifications of the same process that are continuous are indistinguishable. From now on we will always assume that $F(z,r)$ is the continuous version.

1.3.3 Markov property

Theorem 1.3.5 (Markov property)**.** If $D \Subset U$ and G is a Gaussian Free Field on D, then

$$G + H = F,$$

where F is a Gaussian Free Field on U and H is harmonic in D and independent of G. In particular, if we consider two disjoint annuli

$$D(z, r_1) \setminus D(z, r_2) \text{ and } D(w, s_1) \setminus D(w, s_2),$$

then $F(z, r_1) - F(z, r_2)$ and $F(w, s_1) - F(w, s_2)$ are independent.

Proof. We give only a brief sketch of the proof, that can be found in details in [1].

We consider a regular domain $D \subseteq U$ and we use the following decomposition

$$H_0^1(U) = Harm(D) \oplus^{\perp} H_0^1(D),$$

where $H_0^1(D)$ denotes the functions in $H_0^1(D)$ which have been extended to zero outside D and $Harm(D)$ denotes the functions defined on U which are harmonic in D.

Given such decomposition, we recall that the Gaussian Free Field is defined as

$$F(\omega) = \sum G_k(\omega) f_k,$$

where f_k is an orthonormal basis of $H_0^1(U)$. If we choose a basis (ψ_k, λ_k) of $H_0^1(U)$ such that ψ_k is an orthonormal basis of $Harm(D)$ and λ_k is an orthonormal basis of $H_0^1(D)$, we can write

$$F(\omega) = \underbrace{\sum G_k^1(\omega)\psi_k}_{:=H(\omega)} + \underbrace{\sum G_k^2(\omega)\lambda_k}_{:=G(\omega)},$$

where $G(\omega)$ is by definition a Gaussian Free Field on D and $H(\omega)$ is a harmonic function in D (one can show that it converges in the sense of distributions, and since it is harmonic, it is a proper function). Moreover, G and H are independent, since they are defined as series of independent Gaussian random variables.

Finally, if we have two disjoint annuli $D(z, r_1) \setminus D(z, r_2)$ and $D(w, s_1) \setminus D(w, s_2)$, we take the smallest of the two, that without loss of generality we will assume to be $D := D(z, r_1) \setminus D(z, r_2)$ and we use the previous decomposition with this choice of D. It follows that

$$F(z, r_1) - F(z, r_2) = \langle G + H, \eta_{z,r_1} - \eta_{z,r_2} \rangle = \langle G, \eta_{z,r_1} - \eta_{z,r_2} \rangle,$$

in fact the term with H does not contribute, since it corresponds to the difference of two averages of a harmonic function on concentric circles. Moreover

$$F(w, s_1) - F(w, s_2) = \langle F, \eta_{w,s_1} - \eta_{w,s_2} \rangle = \langle H, \eta_{w,s_1} - \eta_{w,s_2} \rangle,$$

since G is zero outside D.

This shows that $F(z, r_1) - F(z, r)$ and $F(w, s_1) - F(w, s)$ are independent. $\square$

1.4 Definition of the Gaussian Free Field in other domains

A lot of different approaches can be taken into account for the construction of the Gaussian Free Field in a bounded domain U with smooth boundary. The construction as a series of independent standard Gaussian random variables is still possible (for further details see [1], [2]).

Other ways to define the Gaussian Free Field are the following ones.

- A stochastic Gaussian process indexed by the set of measures μ compactly supported in U such that

$$\int_{U^2} G_D(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) < +\infty,$$

  (where G_D stands for the Green function of U with Dirichlet boundary conditions). The covariance function is given by

$$Cov\left[F_\mu, F_\nu\right] := \int_{U^2} G_D(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

  (see [1]);

- Construction of the suitable Abstract Wiener Space (see [4]);

- Construction as a Gaussian Hilbert Space, namely as a complete vector space V with

$$V := \left\{\langle F, f\rangle_\nabla\right\}_{f\in H_0^1(U)},$$

  where each element of V is a real Gaussian random variable and the covariances are given by

$$\mathbb{E}\left[\langle F, f\rangle_\nabla \langle F, g\rangle_\nabla\right] = \langle f, g\rangle_\nabla$$

  (see [4]);

- Heuristic definition as the random field whose covariance operator is the Green kernel, namely

$$\mathbb{E}\left[F(x)F(y)\right] = G_D(x,y).$$

It is possible, although quite technical, to show the equivalence among these definitions. Some properties, as the one that follows, are much clearer in one context than in another.

1.5 Thick points

Let us consider a domain U (that can be a bounded subset of $\mathbb{R}^2$ with smooth boundary or the two dimensional torus) and a Gaussian Free Field F defined on U.

Given any nonnegative number a, for every fixed value of ω we define the set of the a-thick points as

$$T(a,U)(\omega) := \left\{ z \in U \text{ s.t. } \lim_{r\to 0^+} \frac{F(z,r)}{\log\frac{1}{r}} = \sqrt{\frac{a}{\pi}} \right\}. \tag{1.5.1}$$

Intuitively, since $F(z,r)$ represents an average of the Gaussian Free Field F around the point z, as r approaches 0 one might expect that $F(z,r)$ approaches the value of $F(z)$. This value, however, is not well defined, because $F(\omega)$ is just a distribution.

Therefore a point z is said to be "a-thick" if around z there is a particular concentration for F, i.e. if $F(z,r)$ behaves asymptotically as $\sqrt{\frac{a}{\pi}} \log \frac{1}{r}$.

The aim of the rest this note is proving the following theorem, due to X. Hu, J. Miler and Y. Peres (see [2]).

Theorem 1.5.1 (X. Hu, J. Miler, Y. Peres)**.**
Almost surely the Hausdorff dimension of $T(a,U)(\omega)$ is $2-a$.

From now on, we will drop for simplicity the complete notation. Therefore we fix the value of the real number a once for all, and then we will write $T(\omega)$ or just T instead of $T(a,U)(\omega)$. Therefore, from now on the letter T will stand only for the set (ω-dependent) of thick points, and in order to avoid confusion we will call our domain U, when needed (even if it might be the torus).

2

Upper bound

Let us fix any positive small number ε. In the whole note C is a constant dependent only on $\varepsilon, \gamma, \zeta$ and independent of n, z. We allow C to change value from line to line. We define $K := \frac{1}{\varepsilon}$ and $r_n := n^{-K}$.

2.1 Step 1: choosing a bigger set

Our aim is to find a bigger set in which the set of thick points is contained. We will therefore show that

$$T \subseteq \hat{T}(\hat{a}),$$

where $\hat{T}(\hat{a})$ is a suitable set that has the form of countable intersections and unions of disks. This leads to a great simplification in computing the Hausdorff dimension.

We begin with this lemma. We give it in the weakest form that is sufficient for all our following estimates, but it can be easily improved by adjusting the value of δ.

Lemma 2.1.1. For every $\zeta > 0$ there exists a constant C such that, if $r \in (r_{n+1}, r_n]$, then

$$|F(z,r) - F(z,r_n)| \leq C(\log n)^{\zeta}$$

for every z.

Proof. We use Theorem 1.3.4 with parameters $\gamma \in \left(0, \frac{1}{2}\right)$, $\delta = \gamma\varepsilon$ on the points

$$\begin{cases} (z,r), \\ (z,r_n), \end{cases}$$

(that eventually satisfy $\frac{r}{r_n} \in \left[\frac{1}{2}, 2\right]$) so that

$$\begin{aligned} |F(z,r) - F(z,r_n)| &\leq M\left(\log\frac{1}{r}\right)^{\zeta}\left(\frac{|(z,r)-(z,r_n)|^{\gamma}}{r^{\gamma+\gamma\varepsilon}}\right) \\ &\leq M \cdot \varepsilon^{-\zeta} \cdot \log(n+1)^{\zeta}\left(\frac{|r_n - r_{n+1}|^{\gamma}}{{r_{n+1}}^{\gamma+\gamma\varepsilon}}\right) \\ &= C \cdot (\log n)^{\zeta}\left(|r_n - r_{n+1}|^{\gamma}\right) \cdot (n+1)^{K(\gamma+\gamma\varepsilon)} \\ &= C \cdot (\log n)^{\zeta}\left(|r_n - r_{n+1}|^{\gamma}\right) \cdot (n+1)^{K\gamma+\gamma}. \end{aligned} \tag{2.1.1}$$

We now use that for a fixed value of K, eventually in n

$$\frac{1}{n^K} - \frac{1}{(n+1)^K} \leq \frac{K+1}{n^{K+1}},$$

and this implies that

$$\left[\frac{1}{n^K} - \frac{1}{(n+1)^K}\right]^\gamma = |r_n - r_{n+1}|^\gamma \leq C\left[\frac{1}{n^{K+1}}\right]^\gamma. \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2) we finally obtain that

$$\begin{aligned}|F(z,r) - F(z,r_n)| &\leq C(\log n)^\zeta \left[\frac{1}{n^{K+1}}\right]^\gamma \cdot (n+1)^{K\gamma+\gamma}\\ &\leq C(\log n)^\zeta \left[\frac{1}{n^{K+1}}\right]^\gamma \cdot (2n)^{K\gamma+\gamma}\\ &= C(\log n)^\zeta.\end{aligned}$$

□

We define

$$T_{\geq}(\omega) := \left\{z \in U \text{ s.t. } \limsup_{r\to 0^+} \frac{|F(z,r)|}{\log\frac{1}{r}} \geq \sqrt{\frac{a}{\pi}}\right\}.$$

It is immediate to see that for every ω

$$T(\omega) \subseteq T_{\geq}(\omega).$$

Lemma 2.1.2 (Reduction to a sequence)**.** It holds that

$$T_{\geq} = \left\{z \in U \text{ s.t. } \limsup_{n\to+\infty} \frac{|F(z,r_n)|}{\log\frac{1}{r_n}} \geq \sqrt{\frac{a}{\pi}}\right\}. \tag{2.1.3}$$

Proof. One inclusion is trivial, so we show the other one. If we suppose that

$$\limsup_{r\to 0+} \frac{|F(z,r)|}{\log\frac{1}{r}} \geq \sqrt{\frac{a}{\pi}},$$

then there exists a sequence $s_h \to 0^+$ such that

$$\frac{|F(z,s_h)|}{\log\frac{1}{s_h}} \geq \sqrt{\frac{a}{\pi}} - \lambda_h,$$

where λ_h goes to zero as h approaches infinity.

We consider the subsequence of r_n, which we call r_{n_h}, such that $s_h \in (r_{n_h+1}, r_{n_h}]$. Using Lemma 2.1.1 with $\zeta < 1$ we obtain that

$$\begin{aligned}|F(z,r_{n_h})| &\geq |F(z,s_h)| - |F(z,r_{n_h}) - F(z,s_h)|\\ &\geq \left(\sqrt{\frac{a}{\pi}} - \lambda_h\right)\log\frac{1}{s_h} - C(\log n_h)^\zeta\\ &= \left(\sqrt{\frac{a}{\pi}} - \lambda_h\right)\log\frac{1}{s_h} - C\left(\frac{1}{K}\log\left(\frac{1}{r_{n_h}}\right)\right)^\zeta,\end{aligned}$$

and this implies that

$$\begin{aligned}
\frac{|F(z,r_{n_h})|}{\log\left(\frac{1}{r_{n_h}}\right)} &\geq \frac{\left(\sqrt{\frac{a}{\pi}}-\lambda_h\right)\log\frac{1}{s_h} - C\left(\log\left(\frac{1}{r_{n_h}}\right)\right)^{\zeta}}{\log\left(\frac{1}{r_{n_h}}\right)} \\
&= \left(\sqrt{\frac{a}{\pi}}-\lambda_h\right)\frac{\log\left(\frac{1}{s_h}\right)}{\log\left(\frac{1}{r_{n_h}}\right)} - \frac{C\left(\log\left(\frac{1}{r_{n_h}}\right)\right)^{\zeta}}{\log\left(\frac{1}{r_{n_h}}\right)} \\
&\geq \sqrt{\frac{a}{\pi}} - \lambda_h - C\log\left(\frac{1}{r_{n_h}}\right)^{\zeta-1} \\
&\geq \sqrt{\frac{a}{\pi}} - o(1),
\end{aligned}$$

where the last inequality follows from the fact that $\zeta < 1$. Therefore we have shown that

$$\limsup_{n\to+\infty}\frac{|F(z,r_n)|}{\log\frac{1}{r_n}} \geq \sqrt{\frac{a}{\pi}},$$

and this concludes the proof of the Lemma. $\square$

We fix p such that $p\gamma-\gamma-\varepsilon := \beta > 0$, therefore such that $p > 1+\frac{\varepsilon}{\gamma}$. When ε will go to zero, $p = p(\varepsilon)$ can be as close to 1 as we want.

Now, for every fixed value of n, we fix any optimal r_n^p net, i.e. a set of points $\left(z_{n_j}\right)_{j\in J_n}$ in U such that

- $\overline{U} \subseteq \bigcup_{j\in J_n} D(z_{n_j}, r_n^p)$,
- any proper subset of J_n has not the same property.

Since $\overline{U}$ is compact, for every p and for every n the set J_n is finite. It is simple to notice that $|J_n| \sim \frac{C}{(r_n^p)^2}$.

We call $b := \sqrt{\frac{a}{\pi}}$ and we choose any $\hat{b} < b$, we call $\mu := b-\hat{b} > 0$ and we define

$$\begin{aligned}
L_n &:= \left\{j \in J_n \text{ s.t. } \left|F\left(z_{n_j}, r_n\right)\right| \geq \hat{b}\log\left(\frac{1}{r_n}\right)\right\}, \\
I(\hat{b}, N_0) &:= \bigcup_{n\geq N_0}\left\{D_{z_{n_j}} \text{ s.t. } j\in L_n\right\}.
\end{aligned}$$

Intuitively, L_n is the set of indexes such that the corresponding center z_{n_j} is behaving like an almost-thick point. The sets $I(\hat{b}, N_0)$ are obviously ordered by inclusion.

Lemma 2.1.3.
For every $\hat{b} < b$ it holds that

$$\left\{z\in U \text{ s.t. } \limsup_{n\to+\infty}\frac{|F(z,r_n)|}{\log(\frac{1}{r_n})} \geq b\right\} \subseteq \bigcap_{N_0\in\mathbb{N}} I(\hat{b}, N_0). \tag{2.1.4}$$

Proof. We use again Theorem 1.3.4 with $\zeta = 1$ to obtain that there exists a constant $M = M(\gamma, \varepsilon, \zeta)$ such that

$$\begin{aligned}\left|F(z, r_n) - F(z_{n_j}, r_n)\right| &\leq M\left(\log\left(\frac{1}{r_n}\right)\right)\left(\frac{\left|z - z_{n_j}\right|^{\gamma}}{{r_n}^{\gamma+\varepsilon}}\right)\\ &\leq M\left(K \log n\right)\left(\frac{{r_n}^{p\gamma}}{{r_n}^{\gamma+\varepsilon}}\right)\\ &= C\left(\log n\right){r_n}^{p\gamma-\gamma-\varepsilon}\\ &= C\left(\log n\right)n^{-K\beta}.\end{aligned}$$

Therefore, if we suppose that

$$\limsup_{n\to+\infty}\frac{|F(z, r_n)|}{\log\left(\frac{1}{r_n}\right)} \geq b,$$

then there exists a sequence n_h such that

$$\frac{|F(z, r_{n_h})|}{\log\left(\frac{1}{r_{n_h}}\right)} \geq b - \lambda_h,$$

where λ_h goes to zero as h approaches infinity.

Now we are interested in the following

$$\begin{aligned}\left|F(z_{n_{hj}}, r_{n_h})\right| &\geq |F(z, r_{n_h})| - \left|F(z, r_{n_h}) - F(z_{n_j}, r_{n_h})\right|\\ &\geq (b - \lambda_h)\log\left(\frac{1}{r_{n_h}}\right) - C\left(\log n_h\right){n_h}^{-K\beta}\\ &= (b - \lambda_h)\log\left(\frac{1}{r_{n_h}}\right) - C\left(\frac{1}{K}\log\left(\frac{1}{r_{n_h}}\right)\right){n_h}^{-K\beta}\\ &= \log\left(\frac{1}{r_{n_h}}\right)\left(b - \lambda_h - C{n_h}^{-K\beta}\right)\\ &\overset{?}{\geq} \hat{b}\cdot\log\left(\frac{1}{r_{n_h}}\right).\end{aligned} \tag{2.1.5}$$

The last inequality is equivalent to

$$\left(b - \lambda_h - C{n_h}^{-K\beta}\right) \overset{?}{\geq} \hat{b},$$

$$\lambda_h + C{n_h}^{-K\beta} \overset{?}{\leq} \mu,$$

$$\lambda_h + C{n_h}^{-K\beta} \overset{?}{\leq} \mu. \tag{2.1.6}$$

The limit of the left-hand side of (2.1.6) is 0, therefore there exists $h_0 = h_0(\gamma, \varepsilon, \zeta, z)$ such that $\lambda_h + C{n_h}^{-K\beta} \leq \mu$ for every $h > h_0$.

Therefore (2.1.6) is eventually true in h, and therefore (2.1.5) holds infinitely many times in n, namely we have proved that

$$\limsup_{n\to+\infty} \frac{|F(z,r_n)|}{\log\left(\frac{1}{r_n}\right)} \geq b, \Longrightarrow \left|F(z_{n_j},r_n)\right| \geq \hat{b}\cdot\log\left(\frac{1}{r_n}\right) \text{ for infinitely many } n$$
$$\Longrightarrow z \in \bigcap_{N_0\in\mathbb{N}} I(\hat{b},N_0),$$

that was exactly the thesis. □

Putting Lemma 2.1.2 and Lemma 2.1.3 together, it follows that for every $\hat{b} < \sqrt{\frac{a}{\pi}}$

$$T \subseteq T_{\geq} \subseteq \bigcap_{N_0\in\mathbb{N}} I(\hat{b},N_0) = \bigcap_{N_0\in\mathbb{N}} \bigcup_{n\geq N_0} \bigcup_{j\in L_n} D(z_{n_j},r_n).$$

2.2 Step 2: proving the upper bound on the bigger set

We define $\eta = \eta(p,\varepsilon) = \eta(\varepsilon)$ so that

$$d := 2 - \pi\hat{b}^2 + \eta > 2p - \pi\hat{b}^2 + \varepsilon$$

and

$$\lim_{\varepsilon\to 0^+} \eta(\varepsilon) = 0.$$

Lemma 2.2.1. For every ε, for every $\delta > 0$

$$\lim_{N_0\to+\infty} \mathbb{E}\left[\mathcal{H}_\delta^{2-\pi\hat{b}^2+\eta}\left(\bigcup_{n\geq N_0}\bigcup_{j\in L_n} D(z_{n_j},r_n)\right)\right] = 0,$$

where $\mathcal{H}_\delta^d$ is as usual

$$\mathcal{H}_\delta^d(E) := \inf\left\{\sum_{i\in I} diam(E_i)^d \,|\, diam(E_i) \leq \delta,\, E \subseteq \bigcup_{i\in I} E_i\right\}.$$

Proof. We use the fact that

$$X \sim N(0,1) \Longrightarrow \mathbb{P}\left(|X| > \lambda\right) \leq C\frac{1}{\lambda}e^{-\frac{\lambda^2}{2}},$$

hence

$$X \sim N(0,\sigma^2) \Longrightarrow \mathbb{P}\left(|X| > b\right) \leq C\frac{\sigma}{b}e^{-\frac{b^2}{2\sigma^2}}.$$

We compute

$$\begin{aligned}
\mathbb{P}(j \in L_n) &= \mathbb{P}\left(\left|F\left(z_{n_j}, r_n\right)\right| \geq \hat{b} \cdot \log\left(\frac{1}{r_n}\right)\right) \\
&= \mathbb{P}\left(\left|N\left(0, \frac{1}{2\pi} \log \frac{1}{r_n}\right)\right| \geq \hat{b} \cdot \log\left(\frac{1}{r_n}\right)\right) \\
&\leq \frac{C}{\hat{b} \cdot \log\left(\frac{1}{r_n}\right)} \sqrt{\frac{1}{2\pi} \log \frac{1}{r_n}} \cdot \exp\left(-\frac{\hat{b}^2 \log^2\left(\frac{1}{r_n}\right)}{2 \cdot \frac{1}{2\pi} \log \frac{1}{r_n}}\right) \\
&\leq C \exp\left(-\frac{\hat{b}^2 \log^2\left(\frac{1}{r_n}\right)}{\frac{1}{\pi} \log \frac{1}{r_n}}\right) \\
&= C e^{-\pi \hat{b}^2 \log\left(\frac{1}{r_n}\right)} \\
&= C \left(\frac{1}{r_n}\right)^{-\pi \hat{b}^2} \\
&= C n^{-K\pi \hat{b}^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}(|L_n|) &= |J_n| \cdot \mathbb{P}(j \in L_n) \\
&\leq C n^{2Kp} \cdot n^{-K\pi \hat{b}^2} \\
&\leq C \cdot n^{2Kp - K\pi \hat{b}^2}.
\end{aligned}$$

Let us estimate

$$\mathbb{E}\left[\mathcal{H}^d_\delta\left(\bigcup_{n \geq N_0} \bigcup_{j \in L_n} D(z_{n_j}, r_n)\right)\right],$$

and finally substitute d with the Hausdorff dimension $2 - \pi \hat{b}^2 + \eta$. If N_0 is big enough, the diameters $2r_n$ are smaller than δ, and therefore for N_0 sufficiently large we have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{H}^d_\delta\left(\bigcup_{n \geq N_0} \bigcup_{j \in L_n} D(z_{n_j}, r_n)\right)\right] &\leq \mathbb{E}\left[\sum_{n \geq N_0} |L_n| \cdot 2{r_n}^d\right] \\
&\leq C \cdot \mathbb{E}\left[\sum_{n \geq N_0} |L_n| \cdot {r_n}^d\right] \\
&= C \cdot \sum_{n \geq N_0} \mathbb{E}\left[|L_n|\right] \cdot n^{-Kd} \\
&\leq C \cdot \sum_{n \geq N_0} n^{2Kp - K\pi \hat{b}^2 - Kd}.
\end{aligned} \tag{2.2.1}$$

Finally we examine the exponent of the right-hand side, and we claim that it is strictly lower than -1. Recalling that $K = \varepsilon^{-1}$, this is equivalent to check that

$$\begin{aligned}
-1 &> -K\pi\hat{b}^2 + 2Kp - Kd, \\
-1 &> -\frac{\pi\hat{b}^2}{\varepsilon} + 2\frac{p}{\varepsilon} - \frac{d}{\varepsilon}, \\
-\varepsilon &> -\pi\hat{b}^2 + 2p - d, \\
d &> 2p - \pi\hat{b}^2 + \varepsilon,
\end{aligned}$$

that is true. Therefore (2.2.1) is the tail of a convergent series, and therefore we have proved the limit we wanted

$$\lim_{N_0 \to +\infty} \mathbb{E}\left[\mathcal{H}_\delta^{2-\pi\hat{b}^2+\eta}\left(\bigcup_{n \geq N_0}\bigcup_{j \in L_n} D(z_{n_j}, r_n)\right)\right] = 0.$$

□

Corollary 2.2.2. Applying Fatou's Lemma we obtain that for every ε, $\delta > 0$, almost surely

$$\liminf_{N_0 \to +\infty} \mathcal{H}_\delta^{2-\pi\hat{b}^2+\eta}\left(\bigcup_{n \geq N_0}\bigcup_{j \in L_n} D(z_{n_j}, r_n)\right) = 0,$$

therefore almost surely

$$\mathcal{H}_\delta^{2-\pi\hat{b}^2+\eta}\left(\bigcap_{N_0 \in \mathbb{N}}\bigcup_{n \geq N_0}\bigcup_{j \in L_n} D(z_{n_j}, r_n)\right) = 0,$$

and therefore, exploiting the previous property with $\delta_n \to 0^+$, we get that almost surely

$$\mathcal{H}^{2-\pi\hat{b}^2+\eta}\left(\bigcap_{N_0 \in \mathbb{N}}\bigcup_{n \geq N_0}\bigcup_{j \in L_n} D(z_{n_j}, r_n)\right) = 0. \tag{2.2.2}$$

2.3 Conclusion

From Step 1 we know that for every $\hat{b} < b$

$$T \subseteq \bigcap_{N_0 \in \mathbb{N}}\bigcup_{n \geq N_0}\bigcup_{j \in L_n} D(z_{n_j}, r_n).$$

Using (2.2.2) we deduce that almost surely

$$\mathcal{H}^{2-\pi\hat{b}^2+\eta}(T) = 0.$$

Since this holds for every $\varepsilon > 0$, letting $\varepsilon \to 0^+$ (so that $\eta(\varepsilon) \to 0$) we get

$$dim_{\mathcal{H}}(T) \leq 2 - \pi\hat{b}^2.$$

Now, letting $\hat{b} \to b^-$ we get

$$dim_{\mathcal{H}}(T) \leq 2 - \pi b^2 = 2 - a,$$

and this concludes the proof of the upper bound for the Hausdorff dimension of the thick points.

3

Lower bound

3.1 Heuristics and general strategy

In order to estimate the Hausdorff dimension from below, we define a subset of the thick points, which we call the perfect points. This set is much simpler to handle, and we will prove that its Hausdorff dimension is greater than or equal to $2 - a$.

We now recall a general theorem of geometric measure theory that will be fundamental to our end.

Definition 3.1.1 (Energy of a measure)**.** Let $U \subseteq \mathbb{R}^d$ and let μ be a measure supported on U. The α-energy of the measure μ is defined as

$$I_\alpha(\mu) := \int_U \int_U \frac{\mathrm{d}\mu(z_1)\,\mathrm{d}\mu(z_2)}{|z_1 - z_2|^\alpha}.$$

Theorem 3.1.2 (Frostman's criterion)**.** Let $U \subseteq \mathbb{R}^d$. Let us assume that there exists a measure μ with these properties:

- $\mu(U) > 0$,
- μ is finite,
- μ has finite α-energy.

Then the Hausdorff dimension of U is at least α.

The proof can be found in [3].

Finally, we state and apply this theorem.

Theorem 3.1.3 (Hewitt-Savage zero one law)**.** Let X_n be a sequence of i.i.d. random variables. Let E be an event of the form

$$E := \left\{\omega \text{ s.t } \mathcal{P}\left(X_1(\omega), \dots\right)\right\},$$

where $\mathcal{P}$ is a suitable property. Let us suppose that, given any finite permutation of the indexes σ, then

$$E = E_\sigma := \left\{\omega \text{ s.t } \mathcal{P}(X_{\sigma(1)}(\omega), \dots)\right\}.$$

Then

$$P(E) = 0 \text{ or } P(E) = 1.$$

We can immediately apply this theorem to the Gaussian Free Field to obtain the following proposition.

Proposition 3.1.4. For every $d \geq 0$ the set

$$\left\{\mathcal{H}^d(T(\omega)) > 0\right\}.$$

has probability zero or one.

Proof. We recall that the Gaussian Free Field is defined as a series of i.i.d. standard Gaussian random variables weighted by suitable coefficients (see Definition 1.2.3). The process $F(z,r)$ has been defined in (1.3.1) as the L^2-limit of $\langle F_n, \eta_{z,r}\rangle$ and the set of thick points has been defined in (1.5.1) as

$$T(\omega) := \left\{z \in U \text{ s.t. } \lim_{r\to 0^+} \frac{F(z,r)}{\log \frac{1}{r}} = \sqrt{\frac{a}{\pi}}\right\}.$$

If we consider a finite permutation of the indexes σ, and we apply again the definition of the Gaussian Free Field, we obtain another object that we call F_σ.

We now want to show that

$$\left\{\omega \text{ s.t. } \mathcal{H}^d(T(\omega)) > 0\right\} = \left\{\omega \text{ s.t. } \mathcal{H}^d(T_\sigma(\omega)) > 0\right\},$$

and for this it is sufficient to show that for every $\omega \in \Omega$ it holds that

$$z \in T(\omega) \iff z \in T_\sigma(\omega).$$

But the last implication is true, because the permutation σ only moves a finite set of indexes, so that the difference between

$$\frac{F(z,r)}{\log \frac{1}{r}} \text{ and } \frac{F_\sigma(z,r)}{\log \frac{1}{r}}$$

is negligible (we are dividing by infinity).

We therefore can apply the Hewitt-Savage zero one law (see Theorem 3.1.3) and deduce that the event $\left\{\mathcal{H}^d(T(\omega)) > 0\right\}$ can only have probability 0 or 1. $\square$

The first step of the proof of the lower bound is the construction the set $P(\omega)$ of perfect points, which is a subset of the thick points.

The second step is proving the lower bound on the Hausdorff dimension on the set of perfect points, and this will be done building a suitable measure τ_ω that satisfies the hypotheses of Theorem 3.1.2

3.2 Definition of perfect points

We define $s_n := \frac{1}{n!}$ and the following events

$$E_m(z) := \left\{\omega \mid \sup_{s\in[s_{m+1},s_m)} \sqrt{2\pi}\left|F(z,s) - F(z,s_m) - \sqrt{\frac{a}{\pi}}\left(\log\left(\frac{1}{s}\right) - \log\left(\frac{1}{s_m}\right)\right)\right| \leq \sqrt{\log\left(\frac{1}{s_{m+1}}\right) - \log\left(\frac{1}{s_m}\right)}\right\}, \tag{3.2.1}$$

$$F_m(z) := \left\{\omega \,|\, \forall s \le s_m \quad \sqrt{2\pi}\left|F(z,s) - F(z,s_m)\right| \le \log\left(\frac{1}{s}\right) - \log\left(\frac{1}{s_m}\right) + 1\right\},$$

and finally

$$E^n(z) := \left(\bigcap_{m=0}^{n} E_m(z)\right) \cap F_{n+1}(z).$$

Let us consider any $\omega \in \Omega$. Given any point $z \in U$, we say that z is n-perfect (with respect to ω) if

$$\omega \in E^n(z).$$

From now on we consider a fixed ω and we often omit the explicit dependence on ω.

We fix a unit square $H \subset U$, and we divide it into $\left(\frac{1}{n!}\right)^2$ little squares of side s_n each. The set of the centers at passage n will be denoted as

$$\hat{c}_n := \left\{z_{n_j}\right\}_{j \in \{1,\dots,(n!)^2\}}.$$

Let C_n be the set of centers that are n-perfect. We denote with $S(z,r)$ the square of center z and side equal to r.

We finally define the set of perfect points as

$$P(\omega) := \bigcap_{k\ge 1} \overline{\bigcup_{n \ge k} \bigcup_{z \in C_n} S(z, s_n)}. \tag{3.2.2}$$

By definition $P(\omega)$ is closed.

Lemma 3.2.1 (Perfect points are thick points)**.** Almost surely

$$P(\omega) \subseteq T(\omega).$$

Proof. Let us fix ω, and let us consider $z \in P(\omega)$. We have to check that $z \in T(\omega)$, namely that

$$\lim_{s \to 0^+} \frac{F(z,s)}{\log \frac{1}{s}} - \sqrt{\frac{a}{\pi}} = 0. \tag{3.2.3}$$

We will prove this statement

$$\lim_{s \to 0^+} \frac{F(z,s) - \sqrt{\frac{a}{\pi}} \log \frac{1}{s} - F(z,s_1) + \sqrt{\frac{a}{\pi}} \log \frac{1}{s_1}}{\log \frac{1}{s}} = 0, \tag{3.2.4}$$

which is equivalent to (3.2.3) because the term

$$\frac{F(z,s_1) + \sqrt{\frac{a}{\pi}} \log \frac{1}{s_1}}{\log \frac{1}{s}}$$

goes to zero.

Let us study the absolute value numerator of (3.2.4), which we call $X = X(s)$,

$$X(s) := \left|F(z,s) - \sqrt{\frac{a}{\pi}} \log \frac{1}{s} - F(z,s_1) + \sqrt{\frac{a}{\pi}} \log \frac{1}{s_1}\right|.$$

By hypothesis, since $z \in P(\omega)$, there exists a sequence $n_k \to +\infty$ and a sequence of centers z_{n_k} such that

- z_{n_k} is an n_k-perfect center,
- $d_{n_k} := |z - z_{n_k}|$ goes to zero.

For each n_k, by triangular inequality it holds that

$$X(s) \leq A + B + C,$$

where

$$\begin{aligned} A(s,k) &:= \left| F(z,s) - F(z_{n_k}, s) \right|, \\ B(s,k) &:= \left| F(z_{n_k}, s_1) - F(z, s_1) \right|, \\ C(s,k) &:= \left| F(z_{n_k}, s) - F(z_{n_k}, s_1) - \sqrt{\frac{a}{\pi}} \left(\log \frac{1}{s} - \log \frac{1}{s_1} \right) \right|. \end{aligned}$$

We now show two sub-steps before concluding the proof.
Step 1:
The following limit is zero.

$$\lim_{s \to 0^+} \left(\lim_{k \to +\infty} \frac{A(s,k) + B(s,k)}{\log \frac{1}{s}} \right) = 0.$$

Proof of Step 1:
We use the Hölder regularity (see Theorem 1.3.4) (the choice of the parameters is not relevant, for example we can choose $\gamma = \frac{1}{3}, \zeta = 1, \delta = \frac{2}{3}$) to gain that

$$A(s,k) \leq C \cdot d_{n_k}^{\gamma} \frac{\log \frac{1}{s}^{\zeta}}{s^{\gamma+\delta}},$$

$$B(s,k) \leq C \cdot d_{n_k}^{\gamma} \frac{\log \frac{1}{s_1}^{\zeta}}{s_1^{\gamma+\delta}}.$$

This implies that for every fixed value of s

$$\lim_{k \to +\infty} \frac{A(s,k) + B(s,k)}{\log \frac{1}{s}} = 0,$$

and therefore

$$\lim_{s \to 0^+} \left(\lim_{k \to +\infty} \frac{A(s,k) + B(s,k)}{\log \frac{1}{s}} \right) = 0,$$

which concludes the proof of Step 1.
Step 2:

Defining

$$\begin{aligned} M := M(s) &= \min \left\{ m \in \mathbb{N} \text{ s.t. } (m+1)! > \frac{1}{s} \right\}, \\ \tilde{f}(s) &:= \sum_{i=1}^{M(s)} \sqrt{\log \frac{1}{s_{i+1}} - \log \frac{1}{s_i}} = \sum_{i=1}^{M(s)} \sqrt{\log(i+1)}, \end{aligned}$$

then the following limit is zero

$$\lim_{s\to 0^+} \frac{\tilde{f}(s)}{\log\frac{1}{s}} = 0.$$

Proof of Step 2:

$$\begin{aligned}
\frac{\tilde{f}(s)}{\log\frac{1}{s}} &= \frac{\sum_{i=1}^{M(s)} \sqrt{\log(i+1)}}{\log\frac{1}{s}} \\
&\geq \frac{\sum_{i=1}^{M(s)} \sqrt{\log(i+1)}}{\log(M(s)!)} \\
&= \frac{\sum_{i=1}^{M(s)} \sqrt{\log(i+1)}}{\sum_{i=1}^{M(s)} \log(i)}.
\end{aligned}$$

As $s \to 0^+$, M(s) $\to +\infty$, and therefore by the Stoltz-Cesaro Theorem we deduce that

$$\lim_{s\to 0^+} \frac{\tilde{f}(s)}{\log\frac{1}{s}} = \lim_{M\to+\infty} \frac{\sum_{i=1}^{M} \sqrt{\log(i+1)}}{\sum_{i=1}^{M} \log(i)} = \lim_{M\to+\infty} \frac{\sqrt{\log(M+1)}}{\log M} = 0,$$

as desired.
Conclusion:
We choose

$$M := M(s) = \min\left\{m \in \mathbb{N} \text{ s.t. } (m+1)! > \frac{1}{s}\right\}.$$

This ensures that

$$s_{M(s)+1} < s \leq s_{M(s)}.$$

Now, we estimate $C(s,k)$ as follows

$$\begin{aligned}
C(s,k) \leq \sum_{i=1}^{M-1} &\left| F(z_{n_k}, s_{i+1}) - F(z_{n_k}, s_i) - \sqrt{\frac{a}{\pi}} \left(\log\frac{1}{s_{i+1}} - \log\frac{1}{s_i} \right) \right| \\
&+ \left| F(z_{n_k}, s) - F(z_{n_k}, s_M) - \sqrt{\frac{a}{\pi}} \left(\log\frac{1}{s} - \log\frac{1}{s_M} \right) \right|.
\end{aligned}$$

If $n_k \geq M(s)+1$, we can use that z_{n_k} is in E_m for every $m = 1, \ldots, n_k$, and therefore z_{n_k} is in E_m for every $m = 1, \ldots, M(s)+1$. With this fact, and using the definition in

(3.2.1), we obtain this stronger estimate on C

$$\begin{aligned}C(s,k) &\leq \sum_{i=1}^{M-1} \sqrt{\log\frac{1}{s_{i+1}} - \log\frac{1}{s_i}} + \sqrt{\log\frac{1}{s_{M+1}} - \log\frac{1}{s_M}} \\ &= \sum_{i=1}^{M(s)} \sqrt{\log\frac{1}{s_{i+1}} - \log\frac{1}{s_i}} \\ &= \tilde{f}(s).\end{aligned}$$

Therefore, if n_k is big enough (i.e. greater than or equal to $M(s)+1$), we estimate $X(s)$ as

$$|X(s)| \leq A(s,k) + B(s,k) + \tilde{f}(s).$$

By the previous steps we know that

$$\lim_{s\to 0^+}\left(\lim_{k\to+\infty}\frac{A(s,k)+B(s,k)}{\log\frac{1}{s}}\right) = 0$$

and

$$\lim_{s\to 0^+}\frac{\tilde{f}(s)}{\log\frac{1}{s}} = 0,$$

therefore we deduce that

$$\lim_{s\to 0^+}\frac{X(s)}{\log\frac{1}{s}} = 0,$$

therefore we have proved (3.2.4), as desired. □

We now need some estimates on the probability of the intersection of $E^n(z)$ and $E^n(w)$, for $z \neq w$. Since we need some kind of independence, we will use the Markov Property.

Lemma 3.2.2 (An estimate on the probability of the intersection of $E^n(z)$ and $E^n(w)$)**.** There exists a constant C such that

$$\forall z \in H, \quad \forall l \in \mathbb{N}, \quad \forall w \in S(z, s_l) \setminus S(z, s_{l+1})$$

it holds that

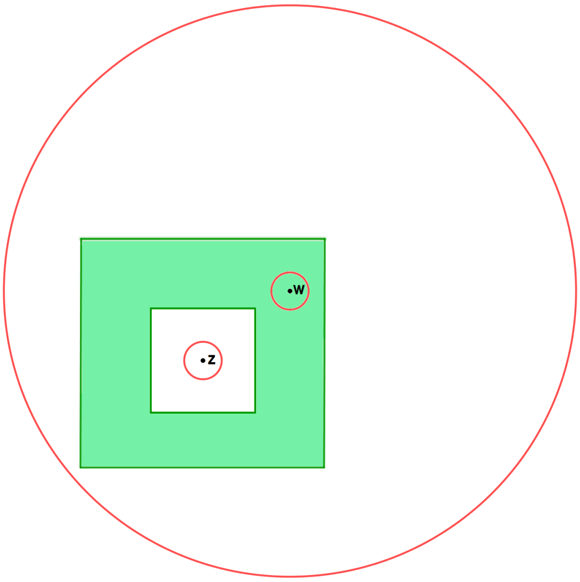
$$P(E^n(z) \cap E^n(w)) \leq C^l \gamma_l^{-a} s_l^{-a} P(E^n(z)) P(E^n(w)),$$

where γ_n is defined by

$$\gamma_n := \prod_{i=1}^{n} e^{\frac{1}{2}\sqrt{(\log i)}}.$$

Proof. Let us fix any $z \in H, l \in \mathbb{N}, w \in S(z, s_l) \setminus S(z, s_{l+1})$. In the picture the "square annulus" $S(z, s_l) \setminus S(z, s_{l+1})$ is colored in green. There are some kind of circles that do not intersect, as the ones drawn in red.

To be more precise, we consider $\partial D(z, s_i)$ and $\partial D(w, s_j)$, and we observe that



- if $i \geq l+2$ and $j \geq l+2$, then $D(w, s_j)$ contains $D(z, s_i)$, therefore $\partial D(z, s_i) \cap \partial D(w, s_j) = \emptyset$,

- if $i \geq l+2$ and $j \leq l-2$, then $D(w, s_j)$ is disjoint from $D(z, s_i)$, therefore $\partial D(z, s_i) \cap \partial D(w, s_j) = \emptyset$.

By Theorem 1.3.5, this implies that

- if $i \geq l+2$ and $j \geq l+2$, the events $E_i(z)$ and $E_j(w)$ are independent,

- if $i \geq l+2$ and $j \leq l-2$, the events $E_i(z)$ and $E_j(w)$ are independent,

- if $i, j \leq n$, the event $F_{n+1}(z)$ is independent both of $E_i(z)$ and of $E_j(w)$,

- if $i, j \leq n$, the event $F_{n+1}(w)$ is independent both of $E_i(z)$ and of $E_j(w)$.

We now estimate

$$\begin{aligned} P\left(E^n(z) \cap E^n(w)\right) &= P\left(\bigcap_{i=1}^{n} E_i(z) \cap \bigcap_{j=1}^{n} E_j(w) \cap F_{n+1}(z) \cap F_{n+1}(w)\right) \\ &= P\left(\bigcap_{i=1}^{n} E_i(z) \cap \bigcap_{j=1}^{n} E_j(w)\right) \cdot P\left(F_{n+1}(z)\right) \cdot P\left(F_{n+1}(w)\right). \quad (3.2.5) \end{aligned}$$

Let us estimate

$$P\left(\bigcap_{i=1}^{n} E_i(z) \cap \bigcap_{j=1}^{n} E_j(w)\right) \leq P\left(\bigcap_{i=l+2}^{n} E_i(z) \cap \bigcap_{\substack{j=1,\dots,n;\\ j\neq l-1,l,l+1}} E_j(w)\right)$$

$$= P\left(\bigcap_{i=l+2}^{n} E_i(z)\right) P\left(\bigcap_{\substack{j=1,\dots,n;\\ j\neq l-1,l,l+1}} E_j(w)\right). \qquad (3.2.6)$$

An application of Girsanov's theorem (which we do not report, but can be found in [2]) allows to estimate the probability of the event $E_i(z)$, which we recall that is of the form

$$E_i(z) := \left\{\sup_{t\in[t_i,t_{i+1})} |B(z,t) - g(t)| \leq \sqrt{\log(i+1)}\right\}.$$

Hence it holds that

$$P\left(E_i(z)\right) \geq \frac{C}{i^a} e^{\frac{a}{2}\sqrt{\log i}},$$

therefore, since the events $E_i(z)$ are independent one from the other, it holds that

$$\begin{aligned} P\left(\bigcap_{i=1}^{l+1} E_i(z)\right) &\geq \prod_{i=1}^{l+1} \frac{C}{i^a} e^{\frac{a}{2}\sqrt{i}} \\ &= \frac{C^{l+1}}{((l+1)!)^a} \prod_{i=1}^{l+1} e^{\frac{\sqrt{i}}{2}a} \\ &= C^{l+1}(s_{l+1})^a \gamma_{l+1}^a. \end{aligned}$$

In an analogous way we obtain that

$$P\left(\bigcap_{j=l-1}^{l+1} E_j(w)\right) \geq \prod_{j=l-1}^{l+1} \frac{C}{j^a} e^{\frac{a}{2}\sqrt{j}}.$$

Therefore

$$P\left(\bigcap_{i=1}^{l+1} E_i(z)\right) P\left(\bigcap_{j=l-1}^{l+1} E_j(w)\right) \geq C^{l+1}(s_{l+1})^a \gamma_{l+1}^a \cdot \prod_{j=l-1}^{l+1} \frac{C}{j^a} e^{\frac{a}{2}\sqrt{j}}. \qquad (3.2.7)$$

We claim that there exists a constant $\hat{C}$ independent of l such that

$$C^{l+1}(s_{l+1})^a \gamma_{l+1}^a \cdot \prod_{j=l-1}^{l+1} \frac{C}{j^a} e^{\frac{a}{2}\sqrt{j}} \geq \hat{C}^l (s_l)^a \gamma_l^a. \qquad (3.2.8)$$

The previous inequality is equivalent to

$$C^{l+4}\left[\frac{1}{(l+1)(l-1)(l)(l+1)}\right]^a e^{\frac{a}{2}\left(\sqrt{l+1}+\sqrt{l-1}+\sqrt{l}+\sqrt{l+1}\right)} \geq \hat{C}^l,$$

therefore we prove a stronger form, that is

$$C^{l+4}\left[\frac{1}{(l+1)}\right]^{4a} \geq \hat{C}^{l}.$$

If we take the l-th root on both sides, we get that

$$\sqrt[l]{C^{l+4}\left[\frac{1}{(l+1)}\right]^{4a}} \geq \hat{C}. \tag{3.2.9}$$

The lim inf of the left-hand side of the previous inequality as l approaches infinity is C, and therefore it is possible to find a constant $\hat{C}$ such that (3.2.9) holds for every $l \in N$, and therefore (3.2.8) is proved.

Putting (3.2.7) and (3.2.8) together it follows that

$$\begin{aligned}
&P\left(\bigcap_{i=1}^{l+1} E_i(z)\right) P\left(\bigcap_{j=l-1}^{l+1} E_j(w)\right) \geq \hat{C}^{l}(s_l)^a\gamma_l^a, \\
&\frac{1}{C'^{l}(s_l)^a\gamma_l^a}\cdot P\left(\bigcap_{i=1}^{l+1} E_i(z)\right) P\left(\bigcap_{j=l-1}^{l+1} E_j(w)\right) \geq 1.
\end{aligned} \tag{3.2.10}$$

Putting (3.2.10) and (3.2.6) together we gain that

$$\begin{aligned}
&P\left(\bigcap_{i=1}^{n} E_i(z) \cap \bigcap_{j=1}^{n} E_j(w)\right) \leq \\
&\leq P\left(\bigcap_{i=l+2}^{n} E_i(z)\right) P\left(\bigcap_{\substack{j=1,\ldots,n;\\ j\neq l-1,l,l+1}} E_j(w)\right)\cdot\frac{1}{C'^{l}(s_l)^a\gamma_l^a}\cdot P\left(\bigcap_{i=1}^{l+1} E_i(z)\right) P\left(\bigcap_{j=l-1}^{l+1} E_j(w)\right) \\
&= \frac{1}{C^{l}(s_l)^a\gamma_l^a} P\left(\bigcap_{i=1}^{n} E_i(z)\right) P\left(\bigcap_{j=1,\ldots,n} E_j(w)\right).
\end{aligned} \tag{3.2.11}$$

Finally, putting (3.2.11) and (3.2.5) together we obtain that

$$\begin{aligned}
P\Big(E^n(z)\cap E^n(w)\Big) &\leq \frac{1}{C^l s_l^a\gamma_l^a} P\left(\bigcap_{i=1}^{n} E_i(z)\right) P\left(\bigcap_{j=1}^{n} E_j(w)\right)\cdot P\left(F_{n+1}(z)\right)\cdot P\left(F_{n+1}(w)\right) \\
&\leq \frac{1}{C^l s_l^a\gamma_l^a} P\Big(E^n(z)\Big) P\Big(E^n(w)\Big),
\end{aligned}$$

as desired. □

3.3 Sequence of random measures

Let $M_n := \left(\frac{1}{n!}\right)^2$ denote the cardinality of all the centers at the n-th passage. Let $p_{n,j} := P(z_{n_j} \in C_n)$. We define this sequence of random measures

$$\begin{aligned}\tau_{\omega,n}(A) :&= \sum_{i=1}^{M_n} \frac{1}{p_{n,i}} \mathbb{1}_{C_n}(z_{n_i}) \int_A \mathbb{1}_{S(z_{n_i},s_n)}(z)\,\mathrm{d}z \\ &= \sum_{\text{centers } z_{n_i} \text{ that are } n\text{-perfect}} \frac{1}{p_{n,i}} \int_{A\cap S(z_{n_i},s_n)} \mathrm{d}z.\end{aligned}$$

We recall that the measure $\tau_{\omega,n}$ depends on ω through the definition of C_n, which is ω-dependent.

The measures $\tau_{\omega,n}$ are **not** supported on $P(\omega)$, but their limit (in the appropriate sense) will. We therefore are interested in proving some uniform bounds.

Lemma 3.3.1. For every value of the constant C, it holds that

$$\sum_{l=1}^{\infty} C^l \gamma_l^{-a} s_l^{2-a} < +\infty, \tag{3.3.1}$$

$$\sum_{l=1}^{\infty} C^l \gamma_l^{-a} (l+1)^{2-a} < +\infty. \tag{3.3.2}$$

Lemma 3.3.2 (Estimates on the sequence of random measures)**.** For every natural number n the following properties hold

$$\begin{aligned}&\mathbb{E}\left[\tau_n(H)\right] = 1,\\ &\mathbb{E}\left[\tau_n(H)^2\right] \le K',\\ &\mathbb{E}\left[I_{2-a}(\tau_n)\right] \le H',\end{aligned}$$

where K', H' are suitable constants that do not depend on n.

Proof. We observe that by definition

$$\tau_{\omega,n}(H) = \sum_{i=1}^{M_n} \frac{1}{p_{n,i}} \mathbb{1}_{C_n}(z_{n_i}) s_n^2.$$

Therefore the first property is a straighforward consequence of the construction, because

$$\begin{aligned}\mathbb{E}\left[\tau_{\omega,n}(H)\right] &= \sum_{i=1}^{M_n} \frac{1}{p_{n,i}} P(z_{n_j} \in C_n) s_n^2 \\ &= \sum_{i=1}^{M_n} s_n^2 \\ &= 1.\end{aligned}$$

We estimate now the second moment

$$
\begin{aligned}
\mathbb{E}\left[\tau_n(H)^2\right] &= s_n^4 \sum_{i,j=1}^{M_n} \frac{1}{p_{n,i}} \frac{1}{p_{n,j}} P(z_{n_i}, z_{n_i} \in C_n) \\
&\le s_n^4 \sum_{i=1}^{M_n} \left(\sum_{l=1}^{\infty} \sum_{z_{n_j} \in S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})} \frac{1}{p_{n,i}} \frac{1}{p_{n,j}} P(z_{n_i}, z_{n_i} \in C_n) \right) \\
&\le K s_n^4 \left| M_n \right| \sum_{l=1}^{\infty} \left(C^l \gamma_l^{-a} s_l^{-a} \frac{s_l^2}{s_n^2} \right) \qquad (3.3.3) \\
&\le K s_n^4 \frac{1}{s_n^2} \sum_{l=1}^{\infty} \left(C^l \gamma_l^{-a} s_l^{-a} \frac{s_l^2}{s_n^2} \right) \\
&\le K \sum_{l=1}^{\infty} \left(C^l \gamma_l^{-a} s_l^{2-a} \right) \\
&\le K', \qquad (3.3.4)
\end{aligned}
$$

where in the passage (3.3.3) we have divided the sum in j in terms corresponding to the "square annuli" $S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})$ and then we have used Lemma 3.2.2 and the fact that in the "square annulus" $S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})$ the number of n-centers is proportional to $\frac{s_l^2}{s_n^2}$.

In the passage (3.3.4) we have applied Lemma 3.3.1, obtaining a constant K' that obviously does not depend on n.

We estimate now the α-energy. As before, we divide the sum in "square annuli" of the form $S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})$, and this allows us both to apply Lemma 3.2.2 and to estimate from above the integrator, as follows

$$
\begin{aligned}
&\mathbb{E}\left[I_\alpha(\tau_n)\right] = \\
&= \sum_{i,j=1}^{M_n} \frac{1}{p_{n,i}} \frac{1}{p_{n,j}} P(z_{n_i}, z_{n_i} \in C_n) \cdot \int_{S(z_{n_i}, s_n)} \int_{S(z_{n_j}, s_n)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{|z_1 - z_2|^\alpha} \\
&= \sum_{i=1}^{M_n} \left(\sum_{l=1}^{\infty} \sum_{z_{n_j} \in S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})} \frac{1}{p_{n,i}} \frac{1}{p_{n,j}} P(z_{n_i}, z_{n_i} \in C_n) \cdot \int_{S(z_{n_i}, s_n)} \int_{S(z_{n_j}, s_n)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{|z_1 - z_2|^\alpha} \right) \\
&\le \sum_{i=1}^{M_n} \left(\sum_{l=1}^{\infty} \sum_{z_{n_j} \in S(z_{n_i}, s_l) \setminus S(z_{n_i}, s_{l+1})} C^l \gamma_l^{-a} s_l^{-a} \cdot \int_{S(z_{n_i}, s_n)} \int_{S(z_{n_j}, s_n)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{|s_{l+1}|^\alpha} \right) \\
&= K \left| M_n \right| \sum_{l=1}^{\infty} \frac{s_l^2}{s_n^2} C^l \gamma_l^{-a} s_l^{-a} s_{l+1}^{-\alpha} s_n^4 \\
&= K \frac{1}{s_n^2} \sum_{l=1}^{\infty} \frac{s_l^2}{s_n^2} C^l \gamma_l^{-a} s_l^{-a} s_{l+1}^{-\alpha} s_n^4 \\
&= K \sum_{l=1}^{\infty} C^l \gamma_l^{-a} (l!)^{a+\alpha-2} (l+1)^\alpha. \qquad (3.3.5)
\end{aligned}
$$

Using (3.3.5) with $\alpha = 2 - a$ and recalling Lemma 3.3.1 this gives

$$\mathbb{E}\left[I_\alpha(\tau_n)\right] \leq K \sum_{l=1}^{\infty} C^l \gamma_l^{-a}(l+1)^{2-a} = H', \tag{3.3.6}$$

where H' is a constant that does not depend on n. □

Proposition 3.3.3 (Definition of G)**.** There exist some constants $d > 0$, $b \in (0,1)$ such that, given

$$G_n := \left\{\omega \text{ s.t. } \tau_{\omega,n}(H) \in \left[b, \frac{1}{b}\right] \text{ and } I_{2-a}(\tau_{\omega,n}) \leq d\right\},$$

$$G := \limsup_n G_n := \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} G_k = \{\omega \text{ s.t. } \omega \in G_{n_k} \text{ for some subsequence } n_k\},$$

then it holds that

$$P(G) > 0.$$

Proof. We use the Paley-Zygmund inequality, which says that

$$P\left(X \geq \lambda \mathbb{E}\left[X\right]\right) \geq (1-\lambda^2)\frac{\mathbb{E}\left[X\right]^2}{\mathbb{E}\left[X^2\right]},$$

and the Markov inequality

$$P\left(X \geq \lambda\right) \leq \frac{\mathbb{E}\left[X\right]}{\lambda},$$

$$P\left(X < \lambda\right) \geq 1 - \frac{\mathbb{E}\left[X\right]}{\lambda}.$$

Therefore

$$P\left(\tau_{\omega,n}(H) \geq \lambda_1\right) \geq (1-\lambda_1^2)\frac{1}{K'},$$
$$P\left(\tau_{\omega,n}(H) < \lambda_2\right) \geq 1 - \frac{1}{\lambda_2},$$
$$P\left(I_{2-a}(\tau_{\omega,n}) < \lambda_3\right) \geq 1 - \frac{H'}{\lambda_3}.$$

Therefore it is possible to choose λ_1, λ_2 and λ_3 such that

$$P\left(\tau_{\omega,n}(H) \geq \lambda_1\right) \geq \frac{1}{2K'} := \mu,$$
$$P\left(\tau_{\omega,n}(H) \leq \frac{\lambda_2}{2}\right) \geq P\left(\tau_{\omega,n}(H) < \lambda_2\right) \geq 1 - \frac{\mu}{3},$$
$$P\left(I_{2-a}(\tau_{\omega,n}) \leq \frac{\lambda_3}{2}\right) \geq P\left(I_{2-a}(\tau_{\omega,n}) < \lambda_3\right) \geq 1 - \frac{\mu}{3},$$

so that

$$P\left(\tau_{\omega,n}(H) \in \left[\lambda_1, \frac{\lambda_2}{2}\right] \text{ and } I_{2-a}(\tau_{\omega,n}) \leq \frac{\lambda_3}{2}\right) \geq \frac{\mu}{3}.$$

Therefore there exist some constants $d > 0$, $b \in (0,1)$, $\beta > 0$ such that

$$P(G_n) \geq \beta \quad \forall n \in \mathbb{N}.$$

Therefore it follows that

$$P(\limsup_n G_n) > 0.$$

In fact, if by contradiction

$$P(\limsup_n G_n) = 0,$$

then

$$P\Big(\omega \text{ s.t. eventually in } n\, \omega \notin G_n\Big) = 1, \tag{3.3.7}$$

$$P\left(\bigcup_{n_0 \in \mathbb{N}} \omega \text{ for every } n \geq n_0\, \omega \notin G_n\right) = 1, \tag{3.3.8}$$

and therefore

$$P\Big(\omega \text{ for every } n \geq N_0\, \omega \notin G_n\Big) \geq 1 - \frac{\beta}{2},$$

for some N_0, but this is absurd, because

$$P(G_{N_0+1}) \geq \beta \quad \forall n \in \mathbb{N}.$$

□

3.4 Limit measure

We therefore restrict to $G \subseteq \Omega$. By definition of G, for every $\omega \in G$ there exists a subsequence n_k such that $\omega \in G_{n_k}$ for all k. Hence the family of measures

$$\{\tau_{\omega,n_k}\}_{k\in\mathbb{N}}$$

admits a weak limit that we call τ_ω such that

- $\tau_\omega(H) \in \left[b, \frac{1}{b}\right]$,
- $I_{2-a}(\tau_\omega) \leq d$,

where the upper bounds follow from the lower semicontinuity of the total mass and the $2-a$ energy with respect to the weak convergence, while the lower bound is a consequence of tightness (H is compact). Moreover, the support of τ_ω is contained in $P(\omega)$, in fact we recall that by definition

$$Supp(\tau_{\omega,n}) \subseteq \bigcup_{n\geq k} \bigcup_{z\in C_n} S(z, s_n),$$

and the set $P(\omega)$ is defined exactly as

$$P(\omega) := \bigcap_{k\geq 1} \overline{\bigcup_{n\geq k} \bigcup_{z\in C_n} S(z, s_n)}$$

(for this passage it is fundamental to have put the closure in the definition of $P(\omega)$). Therefore each τ_ω satisfies the hypotheses of Theorem 3.1.2, and hence

$$\mathcal{H}^{2-a}\left(P(\omega)\right) > 0 \quad \forall \omega \in G,$$

$$P\left(\mathcal{H}^{2-a}\left(P(\omega)\right) > 0\right) \geq P(G) > 0.$$

Recalling Lemma 3.2.1, this implies that

$$P\left(\mathcal{H}^{2-a}(T(\omega)) > 0\right) > 0.$$

Therefore, applying Proposition 3.1.4 it follows that

$$P\left(\mathcal{H}^{2-a}(T(\omega)) > 0\right) = 1,$$

hence almost surely

$$dim_H(T(\omega)) \geq 2 - a.$$

Together with the upper bound, this gives us that almost surely

$$dim_H(T(\omega)) = 2 - a,$$

as desired.

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